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A CLASS OF LOCAL EXPLICIT MANY-KNOT SPLINE INTERPOLATION SCHEME--ETC(U)
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A CLASS OF LOCAL EXPLICIT MANY-KNOT SPLINE INTERPOLATION SCHEMES

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ABSTRACT

The purpose of this paper is to present a new local explicit method for an approximation of real-valued functions defined on intervals. The operators of the form $Qf = \sum \lambda_i f \, q_{i,k}$ are studied under a uniform mesh, where $\{q_{i,k}\}$ comes from a linear combination of B-splines. This paper contains the definition of $\{q_{i,k}\}$, comments on its existence, proof of reproduction of the operator Q for appropriate classes of polynomials, and a note about some applications.

AMS (MOS) Subject Classification: 41A15

Key Words: Many-knot spline function, local, explicit, spline interpolation.

Work Unit Number 3 - Numerical Analysis and Computer Science

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SIGNIFICANCE AND EXPLANATION

The variation diminishing method established by Schoenberg and the quasi-interpolant method developed by de Boor and Fix take the form $Qf = \sum \lambda_i f\, N_{i,k}$ where $\{N_{i,k}\}$ is a sequence of B-splines and $\{\lambda_i\}$ is a sequence of linear functionals. This form is convenient in practices. We would like to keep this form but replace B-spline $N_{i,k}$ with another function $q_{i,k}$, i.e. we consider a different operator $Qf = \sum \lambda_i f\, q_{i,k}$, where $q_{i,k}$ has small support, satisfies $q_{i,k}(j) = \delta_{ij}$, and $\lambda_i f = f(x_i)$. Thus, the operator Q becomes interpolant, and Qf is in a class of the so-called "many-knot" splines. The paper proves that Q reproduces appropriate classes of polynomials. This operator can be used to fit curves or surfaces.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A CLASS OF LOCAL EXPLICIT MANY-KNOT SPLINE INTERPOLATION SCHEMES

D. X. Qi*

As is well known, it is very important to study both theory and application of local spline approximation, such as the variation diminishing method established by Schoenberg, the quasi-interpolant method developed by de Boor and Fix and so on. Those authors studied operators of the form $Qf = \sum_i \lambda_i f\, N_{i,k}$, where $\{N_{i,k}\}$ is a sequence of B-splines and $\{\lambda_i\}$ is a sequence of linear functionals (see [1], [2], [3], [4]).

The purpose of this paper is to present a new method, to get an approximation of real-valued functions defined on intervals. In this method, I use $\{q_{i,k}\}$ to substitute for $\{N_{i,k}\}$ mentioned above as a basic function. The functions $q_{i,k}$ possess the following characteristics: (i) small support (it makes operators of the form $Qf = \sum_i \lambda_i f\, q_{i,k}$ local); (ii) $q_{i,k}(j) = \delta_{ij}$. Here I would only like to discuss how to construct the basic functions $\{q_{i,k}\}$ under $\lambda_i f = f(x_i)$.

Let Δ be a uniform mesh: $a = x_0$, $b = x_n$, $x_i = x_0 + ih$ $(i = 0,1,\ldots,N)$, and additional nodes $x_{-1}, x_{-2}, \ldots$ and $x_{N+1}, x_{N+2}, \ldots$. Let $\hat{S}_p(\Delta,k)$ denote the set of spline functions whose knots are $\{x_i, x_i + \frac{h}{2}\}$. Then $Qf \in \hat{S}_p(\Delta,k)$.

This paper contains the following three parts: (i) definition of a certain basis $\{q_{i,k}\}$ of $\hat{S}_p(\Delta,k)$ and comments on its existence, (ii) proof that Q reproduces approrpriate classes of polynomials, and (iii) a note about some applications.

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1. Construction of $\{q_{i,k}\}$

Let M_k be Schoenberg's centered B-spline of order k on a uniform partition, i.e.,

$$M_k(x) = k[-\frac{k}{2}, -\frac{k-2}{2}, \ldots, \frac{k}{2}](\cdot - x)_+^{k-1} ,$$

and let $I := \{-(k-2), \ldots, k-2\}$. Then the functions

$$M_k(i-\cdot), \quad i \in I$$

are B-splines of order k on the knot sequence $\mathbb{Z} + k/2$, hence independent over the points $I/2$ by the Schoenberg-Whitney Theorem [6] since $M_k(i - i/2) \neq 0$ for $i \in I$. Consequently, the functions

$$M_k(\cdot - j/2), \quad j \in I$$

are independent over I. In particular, there exists exactly one choice of $\gamma := (\gamma_i)_{i \in I}$ so that

$$q_k := \sum_{j \in I} \gamma_j M_k(\cdot - j/2) \tag{1.1}$$

satisfies

$$q_k(i) = \delta_{0i}, \quad \text{all} \quad i \in I . \tag{1.2}$$

Note that $\gamma_{-j} = \gamma_j$ by uniqueness and symmetry (which can be used to simplify the calculation of γ) and that

$$\begin{aligned} 1 = \sum_{i \in I} q_k(i) = \sum_{i \in I} \sum_{j \in I} \gamma_j M_k(i - j/2) = \\ \sum_{j \in I} \gamma_j (\sum_{i \in I} M_k(i - j/2)) = \sum_{j \in I} \gamma_j \end{aligned} \tag{1.3}$$

since $\sum_{i \in I} M_k(i - j/2) = \sum_i M_k(i - j/2) = 1$, all $j \in I$.

Now we define

$$q_{i,k}(\cdot) := q_k(\cdot - i) .$$

The following are the table of coefficients γ and drawings of q_k when $k = 2,3,4$.

k	γ_0	γ_1	γ_2
2	1		
3	2	$-\frac{1}{2}$	
4	$\frac{10}{3}$	$-\frac{4}{3}$	$\frac{1}{6}$

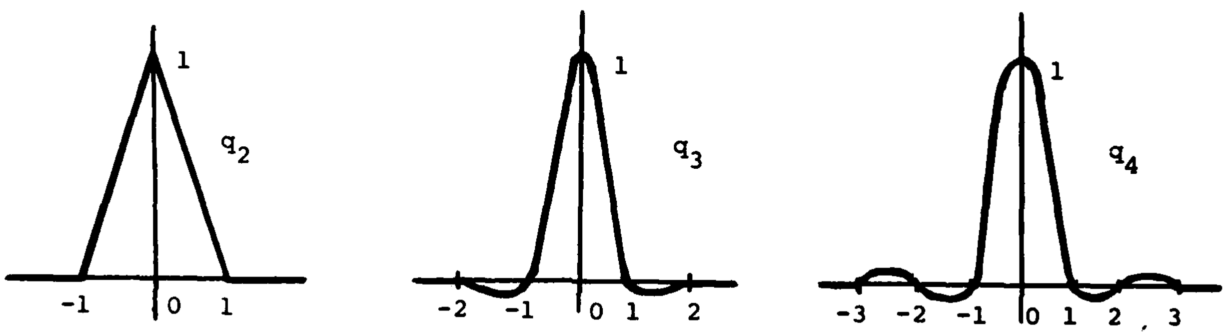

Figure 1

D. X. Qi (1975) has already constructed a class of many-knot spline interpolating functions for solving curve fitting problems ([2], [5]). The main difference between the previous study and the present one is in their basic function. φ_k that appeared in [2] and [5] is not the same as q_k.

2. The interpolation scheme leaves $\mathbb{P}_k$ fixed

In this section I want to prove that Q reproduces certain polynomials. I will use the symbols:

$$\mathrm{sym}_\mu(\alpha_1,\alpha_2,\ldots,\alpha_k) := \sum_{(\nu_1,\ldots,\nu_\mu)} \alpha_{\nu_1}\alpha_{\nu_2}\cdots\alpha_{\nu_\mu} ,$$

$$\nu_j \in \{1,2,\ldots,k\}, \quad \nu_i \neq \nu_j \ (i \neq j) ,$$

$$\mathrm{sym}_0(\cdots) =: \xi_i^{(0)} = 1 ,$$

$$\xi_i^{(\mu)} := \mathrm{sym}_\mu\left(i - \frac{k-1}{2}, i - \frac{k-3}{2},\ldots,i + \frac{k-1}{2}\right)/\binom{k}{\mu} .$$

The letters $\mathbb{P}_k$ denote the set or linear space of all polynomials of order k, i.e., of degree $< k$.

<u>Lemma</u> (simple consequence of Marsden's identity for a uniform partition [4])

$$x^\mu = \sum_i \xi_i^{(\mu)} M_k(x-i), \quad x \in [a,b]$$
$$\mu = 0,1,\ldots,k-1 . \tag{2.1}$$

<u>Theorem 1</u> $Q|_{\mathbb{P}_k} = 1$.

<u>Proof</u> It is enough to prove

$$x^\mu = \sum_i (i)^\mu q_{i,k}(x), \quad x \in [a,b]$$
$$\mu = 0,1,\ldots,k-1 . \tag{2.2}$$

Now we use induction as follows.

Evidently (2.2) holds for $\mu = 0$. Let us assume (2.2) holds throughout $\mu = 0,1,\ldots,m-1$. We will prove it holds for $\mu = m$.

Notice (1.1)

$$q_{i,k}(x) = \sum_{j \in I} \gamma_j M_k\left(x + \frac{j}{2} - i\right)$$

and by lemma

$$\left(x + \frac{j}{2}\right)^\mu = \sum_i \xi_i^{(\mu)} M_k\left(x + \frac{j}{2} - i\right), \quad \mu = 0,1,\ldots,k-1 .$$

Therefore

$$\rho_\mu(x) := \sum_{j\in I} \gamma_j (x + \tfrac{j}{2})^\mu = \sum_i \xi_i^{(\mu)} q_{i,k}(x) \quad . \tag{2.3}$$

Since $\sum_{j\in I} \gamma_j = 1$,

$$\rho_\mu(x) = \sum_{j\in I} \gamma_j \left(\sum_{\nu=0}^{\mu} \binom{\mu}{\nu} x^{\mu-\nu} (\tfrac{j}{2})^\nu \right)$$

$$= \sum_{j\in I} \gamma_j (x^\mu + \sum_{\nu=1}^{\mu} \binom{\mu}{\nu} x^{\mu-\nu} (\tfrac{j}{2})^\nu)$$

$$= x^\mu + \sum_{\nu=1}^{\mu} \binom{\mu}{\nu} (\sum_{j\in I} \gamma_j (\tfrac{j}{2})^\nu) x^{\mu-\nu}$$

$$= x^\mu + \sum_{\nu=1}^{\mu} \binom{\mu}{\nu} \rho_\nu(0) x^{\mu-\nu} \quad . \tag{2.4}$$

By induction hypothesis and (2.3), (2.4),

$$x^m = \rho_m(x) - \sum_{\nu=1}^{m} \binom{m}{\nu} \rho_\nu(0) x^{m-\nu}$$

$$= \sum_i \xi_i^{(m)} q_{i,k}(x) - \sum_{\nu=1}^{m} \binom{m}{\nu} \rho_\nu(0) \sum_i (i)^{m-\nu} q_{i,k}(x)$$

$$= \sum_i (\xi_i^{(m)} - \sum_{\nu=1}^{m} \binom{m}{\nu} \rho_\nu(0)(i)^{m-\nu}) q_{i,k}(x) \quad .$$

Set

$$\eta_i^{(m)} := \xi_i^{(m)} - \sum_{\nu=1}^{m} \binom{m}{\nu} \rho_\nu(0) i^{m-\nu} \quad .$$

Then, from (2.3) and $q_{i,k}(\nu) = \delta_{i\nu}$

$$\rho_j(0) = \sum_i \xi_i^{(j)} q_{i,k}(0) = \xi_0^{(j)} = \frac{\mathrm{sym}_j(-\frac{k-1}{2},\ldots,\frac{k-1}{2})}{\binom{k}{j}} \quad .$$

However

$$\eta_i^{(m)} = \frac{1}{\binom{k}{m}} \mathrm{sym}_m\left(i - \frac{k-1}{2}, i - \frac{k-3}{2}, \dots, i + \frac{k-1}{2}\right) - \sum_{\nu=1}^{m} \binom{m}{\nu} \frac{\mathrm{sym}_\nu\left(-\frac{k-1}{2}, \dots, \frac{k-1}{2}\right)}{\binom{k}{\nu}} i^{m-\nu}$$

$$= \frac{1}{\binom{k}{m}} \left(\mathrm{sym}_m\left(i - \frac{k-1}{2}, \dots, i + \frac{k-1}{2}\right) - \sum_{\nu=1}^{m} \binom{k-\nu}{m-\nu} \mathrm{sym}_\nu\left(-\frac{k-1}{2}, \dots\right) i^{m-\nu}\right)$$

$$= i^m \ .$$

The last identity is gotten by using a well known fact about elementary symmetric function.

From Theorem 1, we can get a result about approximation order.

<u>Theorem 2</u> If $f \in C^{k+1}[a,b]$, then $R_k := f - Qf$

$$\|R_k^{(s)}\|_\infty = \max_{a+(k-1)h \le x \le b-(k-1)h} |R_k^{(s)}(x)| = O(h^{k+1-s})$$

$s = 0,1,\dots,k$.

3. Applications in CAGD

By convention, let $\{P_i\}$ denote a set of ordered points in R^n. We hope to get a curve through $\{P_i\}$. It is known that people in Computer Aided Geometric (CAGD) like and are used to the parametric form. So the curve, as may be imagined, can be represented as follows:

$$Q_k(t) = \sum_j q_k(t-j)P_j \ . \tag{3.1}$$

We can get with ease from this representation and (1.1) in case of $k = 3,4$:

$$Q_3'(j) = \frac{1}{2}(P_{j+1} - P_{j-1}), \quad Q_4'(j) = \frac{4}{3}\left(\frac{P_{j+1}-P_{j-1}}{2}\right) - \frac{1}{3}\left(\frac{P_{j+2}-P_{j-2}}{4}\right)$$

$$Q_4''(j) = 3(P_{j+1} - 2P_j + P_{j-2}) - 2\left(\frac{P_{j+2}-2P_j+P_{j-2}}{4}\right) \text{ etc.}$$

It is simple and useful in CAGD that the interpolating curve is represented by a matrix.

(i) Firstly, we consider a quadratic many-knot spline. Let $t \in [0, \frac{1}{2}]$. We can find

$$(q_3(t+1), q_3(t), q_3(t-1), q_3(t-2)) = (t^2,t,1)\begin{pmatrix} \frac{3}{4} & -\frac{7}{4} & \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.2}$$

$$=: (t^2,t,1)M_3 \quad ,$$

and with the help of symmetry

$$Q_3(t) = \begin{cases} (t^2,t,1)M_3(P_{i-1},P_i,P_{i+1},P_{i+2})^T, & t \in [0, \frac{1}{2}] \quad , \\ ((1-t)^2, 1-t,1)M_3(P_{i+2},P_{i+1},P_i,P_{i-1})^T, & t \in [\frac{1}{2}, 1] \quad . \end{cases}$$

(ii) Secondly we consider a cubic many-knot spline. Let $t \in [0, \frac{1}{2}]$. Then

$$(q_4(t+2),q_4(t+1),\ldots,q_4(t-3)) = (t^3,t^2,t,1) \begin{pmatrix} \frac{7}{36} & -\frac{11}{12} & \frac{14}{9} & -\frac{10}{9} & \frac{1}{4} & \frac{1}{36} \\ -\frac{1}{4} & \frac{3}{2} & -\frac{5}{2} & \frac{3}{2} & -\frac{1}{4} & 0 \\ \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.3}$$

$$=: (t^3,t^2,t,1)M_4 \quad ,$$

and with the help of symmetry

$$Q_4(t) = \begin{cases} (t^3,t^2,t,1)M_4(P_{i-2},P_{i-1},\ldots,P_{i+3})^T, & t \in [0, \frac{1}{2}] \quad , \\ ((1-t)^3, (1-t)^2, 1-t, 1)M_4(P_{i+3}, P_{i+2},\ldots,P_{i-2})^T, & t \in [\frac{1}{2}, 1] \quad . \end{cases}$$

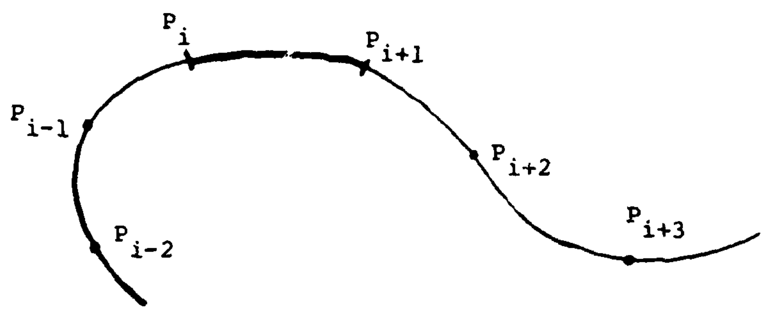

Figure 2

As the parameter t increases from 0 to 1, the segment on the many-knot interpolating spline curve will be traversed from P_i to P_{i+1} (see Figure 2).

If we want to get many-knot spline surfaces when the points $\{P_{i,j}\}$ are given $(i = 0,1,\ldots,N;\ j = 0,1,\ldots,M)$, we could represent the surface as follows:

$$Q_k(u,w) = \sum_{\nu}\sum_{\mu} q_k(u-\nu)q_k(\mu-w)P_{\nu,\mu}$$

$$0 \le \nu \le N, \quad 0 \le w \le M \ ,$$

and this satisfies $Q_k(i,j) = P_{i,j}$.

The representation by matrix for $k = 3$ is:

(I) $Q_3(u,w) = (u^2,u,1)M_3PM_3^T(w^2,w,1)^T, \quad 0 \le u,\ w \le \frac{1}{2}$,

$$P = \begin{pmatrix} P_{i-1,j-1} & P_{i-1,j} & \cdots & P_{i-1,j+2} \\ \cdots & \cdots & \cdots & \cdots \\ P_{i+2,j-1} & \cdots & \cdots & P_{i+2,j+2} \end{pmatrix} = (P_{\nu,\mu})_{\nu=i-1,\ \ \mu=j-1}^{i+2 \qquad j+2} \ .$$

(II) $Q_3(u,w) = ((1-u)^2,1-u,1)M_3PM_3^T(w^2,w,1)^T,\ \frac{1}{2} \le u \le 1,\ 0 \le w \le \frac{1}{2}$,

$$P = (P_{\nu,\mu})_{\nu=i+2,\ \ \mu=j-1}^{i-1 \qquad j+2} \ .$$

(III) $Q_3(u,w) = (u^2,u,1)M_3PM_3^T((1-w)^2,\ 1-w,\ 1)^T,\ 0 \le u \le \frac{1}{2},\ \frac{1}{2} \le w \le 1$,

$$P = (P_{\nu,\mu})_{\nu=i-1,\ \mu=j+2}^{i+2\quad j-1} .$$

(IV)

$Q_3(u,w) = (1-u)^2,\ 1-u,\ 1)M_3PM_3^T((1-w)^2,\ 1-w,\ 1)^T,\ \frac{1}{2} \le u \le 1,\ \frac{1}{2} \le w \le 1$,

$$P = (P_{\nu,\mu})_{\nu=i+2,\ \mu=j+2}^{i-1\quad j-1} .$$

Their figures are shown in Figure 3.

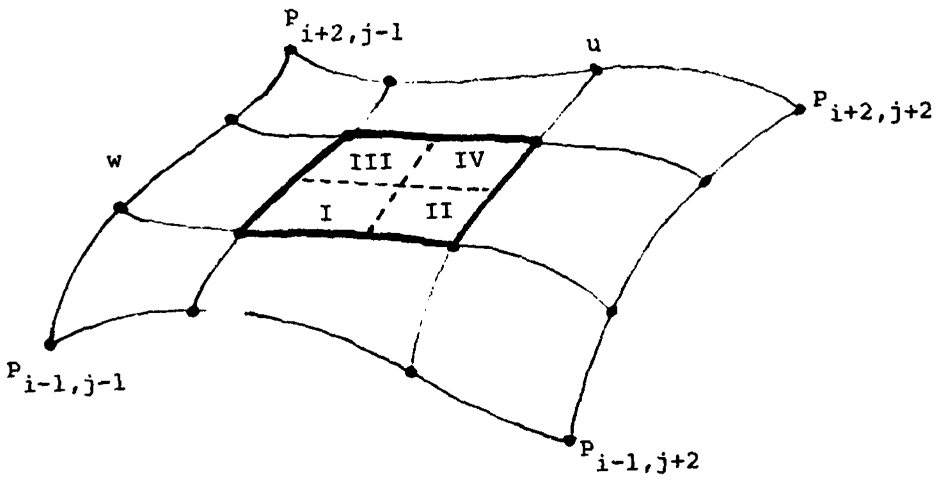

Figure 3

In the case of $k = 4$ the representation and figures can be given as follows:

(I) $Q_4(u,w) = (u^3,u^2,u,1)M_4PM_4^T(w^3,w^2,w,1)^T,\ 0 \le u,\ w \le \frac{1}{2}$,

$$P = (P_{\nu,\mu})_{\nu=i-2,\ \mu=j-2}^{i+3\quad j+3} .$$

(II)

$$Q_4(u,w) = ((1-u)^3,\ (1-u)^2,\ 1-u,\ 1)M_4PM_4^T(w^3,w^2,w,1)^T,\ \frac{1}{2} < u < 1,\ 0 < w < \frac{1}{2}\ ,$$

$$P = (P_{\nu,\mu})_{\nu=i+3,\ \mu=j-2}^{i-2\quad j+3}\ .$$

(III)

$$Q_4(u,w) = (u^3,u^2,u,1)M_4PM_4^T((1-w)^3,(1-w)^2,1-w,1)^T,\ 0 < u < \frac{1}{2},\ \frac{1}{2} < w < 1\ ,$$

$$P = (P_{\nu,\mu})_{\nu=i-2,\ \mu=j+3}^{i+3\quad j-2}\ .$$

(IV)

$$Q_4(u,w) = ((1-u)^3,(1-u)^2,1-u,1)M_4PM_4^T((1-w)^3,(1-w)^2,1-w,1)^T,\ \frac{1}{2} < u,\ w < 1\ ,$$

$$P = (P_{\nu,\mu})_{\nu=i+3,\ \mu=j+3}^{i-2\quad j-2}\ .$$

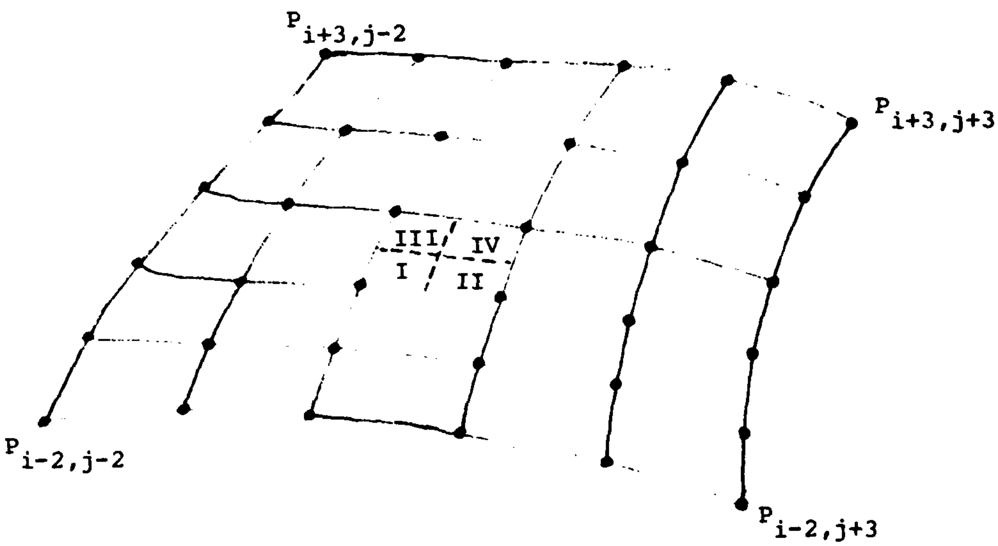

Figure 4

Acknowledgement

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The purpose of this paper is to present a new local explicit method for an approximation of real-valued functions defined on intervals. The operators of the form $Qf = \sum \lambda_i f q_{i,k}$ are studied under a uniform mesh, where $\{q_{i,k}\}$ comes from a linear combination of B-splines. This paper contains the definition of $\{q_{i,k}\}$, comments on its existence, proof of reproduction of the operator Q for appropriate classes of polynomials, and a note about some applications.

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